"A Study of Fractional Calculus Operators Associated with Generalized Functions"

A
Synopsis

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Introduction

Theory of special functions has a long and varied history with immense literature due to their applications in solving various problems arising in physical, biological and engineering sciences. Special functions have an origin in the solution of partial differential equations satisfying certain prescribed conditions. At present special functions are defined in several ways notably by power series, generating functions, infinite products, integrals, difference equations, trigonometric or orthogonal function series. Eminent mathematicians notably Euler, Legendre, Gauss, Jacobi, Weierstrass, Kummer, Riemann, Ramanujan worked hard to develop special functions like Bessel functions, Whittaker functions, Gauss hypergeometric function and the polynomials that go by the names of Jacobi, Legendre, Laguerre, Hermite etc. The Gaussian hyper geometric function ${}_pF_q$ and its special cases are commonly used in applied mathematics and mathematical physics. Since ${}_pF_q$ diverges for $p > q + 1$, in an attempt to give meaning to it in this case , MacRobrtand Meijer introduced the E-function and the G-function[2], respectively. Though these functions are quite general in character, a number of special functions like Wright's generalized hypergeometric function[45], generalized parabolic cylinder function, Mittag-Leffler's function, and several other functions, do not form their special cases.

Fractional Calculus is a field of mathematic study that grow out of the traditional definitions of the calculus integral and derivatives operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definition can clearly become confused when exponents of non integer value.

The fractional derivatives (and fractional integral) of special function of one and more variable is important such as in the evolution of series and integrals the derivation of generating function and the solution of the integral equation motivated by these and many other avenues of applications the fractional calculus operator ${}_{\alpha}D_x^{\mu}$ is much used in the theory of special function of one and more variables.The fractional derivative, extension of the ordinary derivative to an non integral value of the order, is of immense utility in finding the solution of ordinary, partial and integral equations, as well as in other contexts. Fractional derivatives for conventional functions were introduced long before the systematic development of the generalized function. Indeed one of the more creditable

aspects of the new generalized functional theory would seem to be the fractional derivatives as well as hypergeometric function.

A Fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is: Can the meaning of derivatives of integral order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any number irrational, fractional or complex.Leibnitz invented the above notation. It was naive play with symbols that prompted L'Hospital to ask Leibnitz about the possibility that n be a fraction. "What if n be ½? ", asked L'Hospital. Leibnitz in 1695, replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." In 1697, Leibnitz, referring to Wallis's infinite product for π/2, used the notation $d^{½} y$ and stated that differential calculus might have been used to achieve the same result.

In the past century, many authors have generalized H-function. In a recent paper, Saxena et al. [16] have introduced a generalization of Saxena's I-function [22]. This is also a generalization of Fox's H-function. Saxena and Pogany [17] have studied fractional integral formulae for the Aleph function. Sudland et al. [42] studied the generalized fractional driftless Fokker-Planck equation with power law coefficient. As a result a special function was found, which a particular case of the Aleph function is. The results obtained by the authors serve as the key formulas for numerous potentially useful special functions of Science, Engineering and Technology scattered in the literature.

The integral representations involving the product of the Aleph function with exponential function, Gauss hypergeometric function and H-function has been obtained by Sharma [24] and some special cases of the established results are also discussed. The results obtained are useful where the I-function occurs naturally.

Objectives

We propose to study in the following areas:

1. To solve fractional kinetic and diffusion equations involving various fractional integral and differential operators using integral transforms.

2. The study of q- analogues of various integral transforms such as q- Laplace transform, q- Mellin transform, q- Fourier transform, q- Henkel transform etc. and find out their possible new applications.

3. To study of q-analogues on various generalization of fractional differential and integral operators.

Importance of Study

It is expected that most of the results which will obtain in this survey may find applications in the solution of certain fractional order differential and integral equations arising problems of physical science, engineering and statistics.

Review of Literature

A Fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is: Can the meaning of

derivatives of integral order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any number irrational, fractional or complex. Leibnitz invented the above notation. It was naive play with symbols that prompted L'Hospital to ask Leibnitz about the possibility that n be a fraction. "What if n be ½? ", asked L'Hospital. Leibnitz in 1695, replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." In 1697, Leibnitz, referring to Wallis's infinite product for $\pi/2$, used the notation $d^{1/2} y$ and stated that differential calculus might have been used to achieve the same result.

The results derived one of the general character and give rise to a number of known results in the theory of multiindex Mittag-Leffler functions. Shukla and Prajapti [35] investigated a recurrence relation and integral representation of generalized Mittag-Leffler function. Saxena et al. studied generalized Mittag-Leffler function introduced by Saxena and Nishimota [20] integral representations of generalized Mittag-Leffler function are derived. The result obtained provides new result for the Mittag-Leffler functions studies by Shrivastva and Tomovski [41] and Shukla and Prajapati [35]. Sharma et al.[32] introduced the generalized M-series[33] and investigated the fractional integration and differentiation of that series. Sharma and Dhakar [25] introduced a new K_2 - function and derived the relations that exist between the K_2 -function and the operators of Riemann- Liouville fractional integrals and derivatives. Sharma [28] derived the solution of generalized fractional kinetic equation. Results obtain in a compact form in terms of K_4 -function. Sharma [28] introduced a new function namely K_4 -function and demonstrated how K_4 -function is closely related to another special functions. The differintegration of that function is also investigated. Sharma investigated the generalized fractional integration of the generalized M-series and some results derived by Saxena and Siogo [18]. Sharma et al.[33] studied the derivation of certain transformation formulae for the basic hypergeometric functions as an application of fractional q-derivative. Sharma [26] derived a set of generalized fractional Volterra type integral equation involving K_4 -function [28].

Methodology

In the present study, it is proposed:

1. To investigate the fractional calculus operators of the generalized Mittag-Leffler type functions
2. To derive the recurrence relations and integral representations of the generalized Mittag-Leffler type functions.
3. To generalize the functions of fractional calculus and derive the relations that exists among these functions and fractional calculus operators.
4. To use the integral transforms for finding the solutions of the fractional differential and integral equations.
5. To find the solution of fractional differential and integral equations arising in physical, biological and engineering sciences.
6. To deduce the results for the basic hypergeometric functions as an application of fractional q-Calculus.

Some of the important fractional calculus operators are as follows:

Riemann- Liouville left-sided fractional integral of order α

$$ {}_{a}I_{x}^{\alpha} f(x) = {}_{a}D_{x}^{-\alpha} f(s) = I_{a+}^{a} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad x > a \tag{1} $$

Riemann- Liouville right-sided fractional integral of order α

$$ I_{x\,b}^{\alpha} f(x) = D_{xb}^{-\alpha} f(s) = I_{b-}^{a} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad x < b \tag{2} $$

Riemann- Liouville left-sided fractional derivative of order α

$$ {}_{a}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \frac{f(t)\,dt}{(x-t)^{\alpha-n+1}}. \quad (n = [\alpha] + 1) \tag{3} $$

Where $[\alpha]$ denotes the integral part of α.

Riemann- Liouville right-sided fractional derivative of order α

$$ {}_{x}D_{b}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^{n} \int_{x}^{b} \frac{f(t)\,dt}{(x-t)^{\alpha-n+1}}, \quad (n = [\alpha] + 1) \tag{4} $$

Where $[\alpha]$ denotes the integral part of α.

Caputo fractional derivative

$$ {}_{0}^{c}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{f(t)\,dt}{(x-t)^{\alpha}}, \tag{5} $$

Where $0 < \alpha < 1$.

Some integral transforms which will be used in the research work:

Laplace transform

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$ F(s) = L\{f(t)\} = \int_{0}^{\infty} e^{-st} F(t)\,dt \tag{6} $$

$$s = \sigma + i\omega$$

The parameter *s* is a complex number with real numbers σ and ω

Sumudu transform

Watugala [43] introduced the Sumudu transform which is defined over the set of Functions

$$A = \{ f(t) : \exists M, \tau_1, \tau_2 > 0, f(t) < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty)\},$$

$$S\{ f(t)\} = G(s) = \int_0^\infty e^{-t} f(st)\, dt,\ s \in (-\tau_1, \tau_2).$$

(7)

Watugala [85] first advocated the transform as an alternative to the standard Laplace Transform, and gave it the name Sumudu transform.

Fourier transform

The continuous Fourier transform is equivalent to evaluating the bilateral Laplace transform with imaginary argument $s = i\omega$

$$f(\omega) = F\{ f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t}\ f(t)\, dt$$

(8)

This expression excludes the scaling factor $1/\sqrt{2\pi}$, which is often included in definitions of the Fourier transform

Mellin transform

Let f (t) be locally Lebesgue integrable over (0, ∞). The Mellin transform of f (t) is defined by

$$M[f](s) = M\{ f(t)\} = \int_0^\infty t^{s-1} f(t) dt.$$

(9)

Some Special Functions of Fractional Calculus are given below:

Mittag-Leffler functions

Mittag-Leffler [5, 6] in terms of the power series was defined as

$$E_\alpha(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + 1)}, \qquad (\alpha > 0)$$

(10)

$$E_{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + \beta)}, \quad (\alpha, \beta > 0)$$

(11)

A generalization of (10) in the following form has been studied by several authors notably by Mittag-Leffler [5,6],Wiman[44]. A generalization of $E_{\alpha,\beta}$ given by Prabhakar [10] in the form

$$E_{\alpha,\beta}^{\gamma}(t)=\sum_{k=0}^{\infty}\frac{(\gamma)_k\ t^k}{k!\ \Gamma(\alpha k+\beta)}, \tag{12}$$

Where α , β ,$\gamma > 0$ And $\Re(\alpha) > 0$.

In 2007, Shukla and Prajapati[35] introduced the function $E_{\alpha,\beta}^{\gamma,q}(t)$ which is defined for

α , β ,$\gamma \in C$ and $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $q \in (0,1) \cup N$ as

$$E_{\alpha,\beta}^{\gamma,q}(t)=\sum_{k=0}^{\infty}\frac{(\gamma)_{qk}\ t^k}{k!\ \Gamma(\alpha k+\beta)} \tag{13}$$

Where

$$(\gamma)_{qk}=\frac{\Gamma(\gamma+qk)}{\Gamma k}$$

Denotes the generalized Pochhammer symbol (Rainville[11]).

M-Series

The M-series introduced by Sharma [34] and is given by

$$\begin{aligned}{}_p^{\alpha}M_q(t)&={}_p^{\alpha}M_q(a_1,a_2,\ldots a_p;b_1,b_2,\ldots b_q;t)\\&=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b)_{1k}(b)_{2k}\ldots(b)_{qk}}\frac{t^k}{\Gamma(\alpha k+1)}\end{aligned} \tag{14}$$

Here $\alpha \in C$ $\Re(\alpha) > 0$ and $(a_i)_k ; (b_j)_k$ are the Pochammer symbols. Further details of this function are given by Sharma [34].

Generalized M-Series

Generalized M-series introduced by Sharma and Jain [33] and is given by

$$\begin{aligned}{}_p^{\alpha,\beta}M_q(t)&={}_p^{\alpha,\beta}M_q(a_1,a_2,\ldots a_p;b_1,b_2,\ldots b_q;t)\\&=\sum_{k=0}^{\infty}\frac{(a)_{1k}(a)_{2k}\ldots(a)_{pk}}{(b)_{1k}(b)_{2k}\ldots(b)_{qk}}\frac{t^k}{\Gamma(\alpha k+\beta)}\end{aligned} \tag{15}$$

Here $\alpha, \beta \in C\ \Re(\alpha) > 0, \Re(\beta) > 0$ and $(a_i)_k ; (b_j)_k$ are the Pochammer symbols.

Further details of this function are given by [33].

K4- function

The K_4 -function introduced by Sharma [28] and is defined as follows

$$K_{4}^{(\alpha,\beta,\gamma),(a,c):(r;s)}(a_1,....,a_r ; b_1,....,b_s : x) = K_{4}^{(\alpha,\beta,\gamma),(a,c):(r;s)}(x)$$
$$= \sum_{n=0}^{\infty} \frac{(a)_{1n}...(a_r)_n}{(b)_{1n}.....(b)_n} \frac{(\gamma)_n a^n (x-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)} \tag{16}$$

where $R(\alpha\gamma - \beta) > 0$ and $(a_i)_n$ (i=1,2,...,r) and $(b_j)_n$ (j=1,2,...,s) are the Pochhammer symbols. Further details of this function are given by [28].

K-Function

A new generalization of above functions was defined by Sharma [30] as

$${}_pK_q^{\alpha,\beta;\gamma}(a_1,\ldots,a_p; b_1,...,b_q; x) = {}_pK_q^{\alpha,\beta;\gamma}(x)$$
$$= \sum_{r=0}^{\infty} \frac{(a_1)_r...(a_p)_r}{(b_1)_r...(b_q)_r} \frac{(\gamma)_r x^r}{r!\Gamma(\alpha r+\beta)} \tag{17}$$

where $\alpha, \beta, \gamma \in C$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this function is given in [30] and some new properties of that function are recently obtained by Sharma [31]. Throughout this paper, we need the following well-known facts and rules.

Beta transform (Sneddon [38]):

$$B\{f(x);a,b\} = \int_0^1 x^{a-1}(1-x)^{b-1} f(x)dx,\ \mathrm{Re}(a) > 0, \mathrm{Re}(b > 0. \tag{18}$$

Laplace transforms (Sneddon [38]):

$$L\{f(x);s\} = \int_0^{\infty} e^{-sx} f(x)dx,\ \mathrm{Re}(s) > 0. \tag{19}$$

Convolution theorem of Laplace transforms (Fenney et al. [1]):

$$L_{-1}\{f * g\}(s) = \int_0^t f(t-\zeta)g(\zeta)d\zeta .$$

(20)

Mellin transform (Sneddon [38]):

$$M\{f(x);s\} = \int_0^\infty z_s^{-}{}_1 f(z)dz,\ \mathrm{Re}(s) > 0.$$

(21)

And its inverse is given by

$$f(z) = M^{-1}\{f^*(s); z\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s} f^*(s)ds,\ c \in R.$$

(22)

Whittaker transforms (Whittaker and Watson (1952)):

$$\int_0^\infty e^{-t/2} t^{v-1} W_{\lambda,\mu}(t)dt = \frac{\Gamma(1/2+\mu+v)\Gamma(1/2-\mu+v)\delta)}{\Gamma(1-\lambda+v)}$$

(23)

Where $\mathrm{Re}(\mu \pm v) > -1/2$ and $W_{\lambda,\mu}(t)$ is the Whittaker confluent hyper geometric function.

In 1961, Charles Fox [2] introduced a function which is more general in than the Meijer's G-function and this function is well known in the literature of special functions as Fox's H-function. The function is defined and presented by means of the following Mellin-Barnes type contour integral:

$$H(z) = {}_{H_{p,q}}^{m,n}[z] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_L \theta(s) z^s\, ds,$$

(24)

Where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}.$$

(25)

An account of the convergence conditions for this integral can be found in the paper by Fox [2]. The well-known G-function [2] is given by

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_j,1)_{1,p}\\(b_j,1)_{1,q}\end{matrix}\right] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_1,1),\ldots(,a_p,1)\\(b_1,1),\ldots(,b_q,1)\end{matrix}\right] = G_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}1 & p\\ a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds$$

(26)

The Mac Robert's E-function [2] is defined as

$$G^{q}_{p,0,q}\left[z \middle| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right] = E[q; b_j : p; a_j : z]$$

(27)

The I-function which is more general than the Fox's H-function, defined by Saxena [22], by means of the following Mellin-Barnes type contour integral:

$$I^{m,n}_{\ldots} = I^{m,n}_{\ldots}\left[\begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,\ p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right] \frac{1}{2\pi i}\int_L \ldots \, s \ldots$$

(28)

Where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r}\left\{ \prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} + \beta_{ji} s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji} - \alpha_{ji} s)\right\}}.$$

(29)

For details regarding existence conditions and various parameter restrictions of I-function we may refer [22].

Aleph function is the most generalized special function, numerous special cases with potentially useful transcendental functions such as Mittag-Leffler function, Bessel functions, Whittaker functions, hypergeometric functions, generalized hypergeometric function, Meijer's G-function, Fox-Wright function, Fox's H-function and I-function which is recently introduced by Saxena et al.[16], by means of the following Mellin-Barnes type contour integral:

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i,\pi:r}[z] = \aleph^{m,n}_{p_i,q_i,\pi:r}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,n};[\pi(a_j,\alpha_j)]_{n+1,\ p_i} \\ (b_j,\beta_j)_{1,m};[\pi(b_j,\beta_j)]_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\pi:r}(s) z^{-s}\, ds,$$

(30)

For all $z \neq 0$, where $\omega = \sqrt{-1}$ and

$$\Omega_{p_i,q_i,\pi:r}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^{r}\pi_i\left\{\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\right\}}.$$

(31)

An account of the convergence conditions for the above integral can be found in the paper by Saxena and Pogány [16,17].

Two of the most commonly encountered tools in the theory and applications of fractional calculus are provided by the Riemann-Liouville and the Weyl operators which are respectively defined by Srivastava et al.[41]

$$R_x^{\delta}(f(x)) = \frac{1}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1} f(t)dt,\ \mathrm{Re}(\delta)>0$$
$$= \frac{d^n}{dx^n} R^{\delta+n}(f(x)),\ -n < \mathrm{Re}(\delta) \geq 0;\ n \in N$$

(32)

And

$$W^{\delta}{}_x(f(x)) = \Gamma\left(\frac{1}{\delta}\right)\int_x^{\infty} (t-x)^{\delta-1} f(t)dt,\ \mathrm{Re}(\delta)>0$$
$$= \frac{d^n}{dx^n} W^{\delta+n}(f(x)),\ -n < \mathrm{Re}(\delta) \geq 0;\ n \in N$$

(33)

Provided that the defining integrals exist.

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